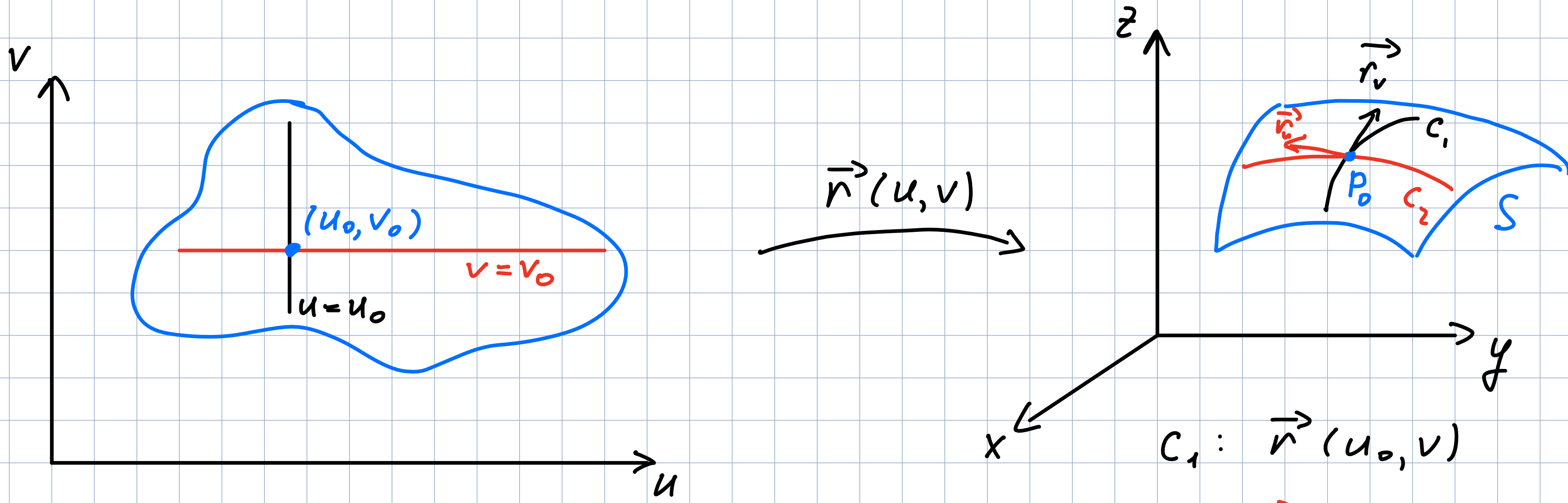
Tangent plane to a parametric surface



$C_1: \ \vec{r}(u_0, v)$

$C_2: \ \vec{r}(u, v_0)$

$\overrightarrow{OP_0} = \vec{r}(u_0, v_0)$

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u_0,v_0),\ \frac{\partial y}{\partial v}(u_0,v_0),\ \frac{\partial z}{\partial v}(u_0,v_0) \right\rangle$$ - tangent vector for C_1 at P_0

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}(u_0,v_0),\ \frac{\partial y}{\partial u}(u_0,v_0),\ \frac{\partial z}{\partial u}(u_0,v_0) \right\rangle$$ - tangent vector for C_2 at P_0

Assume $\vec{r}_u \times \vec{r}_v \neq 0$.
(then S is "smooth")

Then the tangent plane at P_0 has normal vector $\vec{r}_u \times \vec{r}_v$.

Ex: $\begin{cases} x = u^2 \\ y = v^2 \\ z = u + 2v \end{cases}$ - parametric surface. Find the tangent plane at $P_0(1, 1, 3)$.

Sol: 1) $\vec{r}(u,v) = \langle u^2, v^2, u+2v \rangle$

$\vec{r}(u_0, v_0) = \langle 1, 1, 3 \rangle \Rightarrow$

P_0 corresponds to $u_0 = 1$, $v_0 = 1$.

2) $\vec{r}_u = \langle 2u, 0, 1 \rangle$

$\vec{r}_v = \langle 0, 2v, 2 \rangle$

3) At $u_0 = v_0 = 1$:

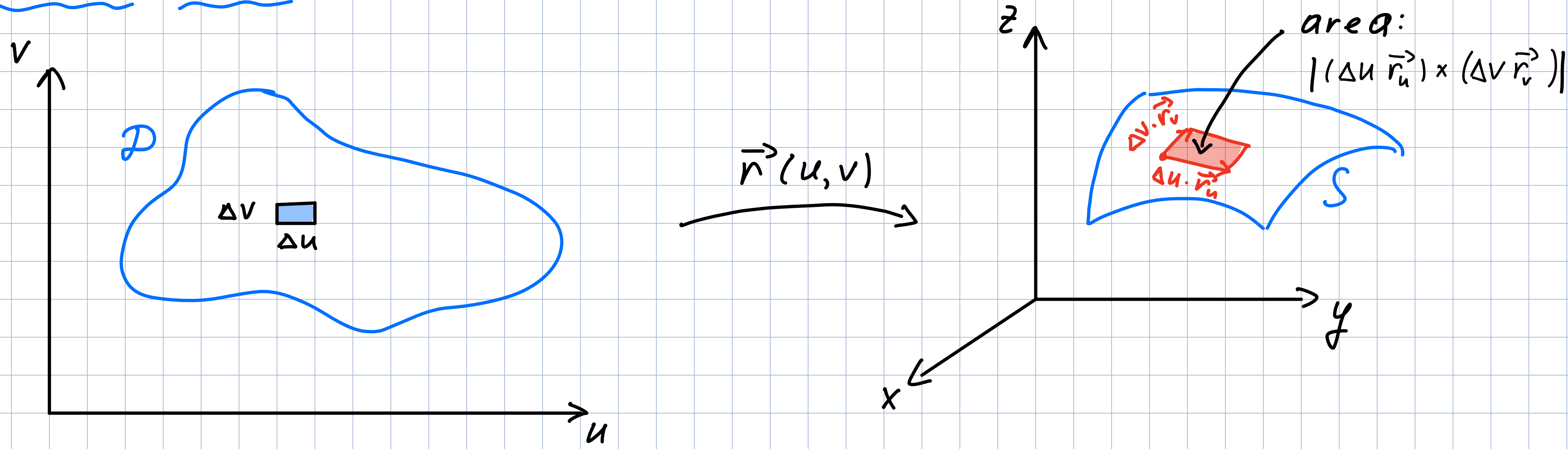
$\vec{r}_u(1,1) = \langle 2, 0, 1 \rangle \qquad \vec{r}_v(1,1) = \langle 0, 2, 2 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \langle 0\cdot 2 - 1\cdot 2,\ 1\cdot 0 - 2\cdot 2,\ 2\cdot 2 - 0\cdot 0 \rangle = \langle -2, -4, 4 \rangle$$

4) equation of tangent plane: $-2(x-1) - 4(y-1) + 4(z-3) = 0$

or $-2x - 4y + 4z - 6 = 0$

Surface area:



Surface area of S:
(a smooth parametric surface)

$$\boxed{\text{Area}(S) = \iint_{D} |\vec{r}_u \times \vec{r}_v|\, dA\,,}$$

where $\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle,$

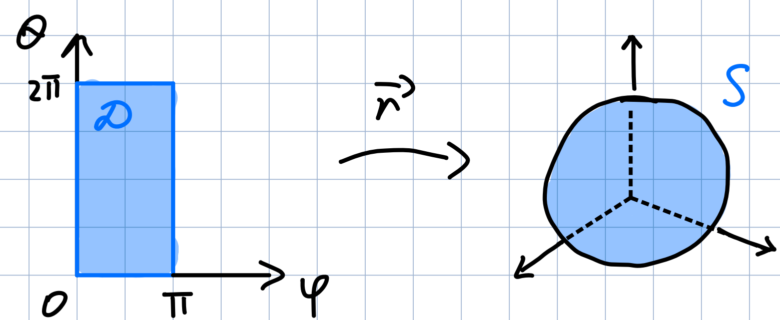
$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$

Rmk: We assume that S is covered once when (u,v) ranges throughout D.

Ex: Find a surface area of a sphere of radius a.

Sol: $\vec{r}(\varphi,\theta) = \langle x(\varphi,\theta),\, y(\varphi,\theta),\, z(\varphi,\theta)\rangle$

$$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases} \qquad \mathcal{D}: \begin{cases} 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$



$$\vec{r}_\varphi = \langle a\cos\varphi\cos\theta,\, a\cos\varphi\sin\theta,\, -a\sin\varphi\rangle$$

$$\vec{r}_\theta = \langle -a\sin\varphi\sin\theta,\, a\sin\varphi\cos\theta,\, 0\rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \langle a^2\sin^2\varphi\cos\theta,\, a^2\sin^2\varphi\sin\theta,\, a^2\sin\varphi\cos\varphi\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}\rangle$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = a^2\sin\varphi\sqrt{\underbrace{\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta}_{\sin^2\varphi} + \cos^2\varphi} = a^2\sin\varphi$$

$$\text{Area}(S) = \int_0^{\pi}\int_0^{2\pi} a^2\sin\varphi\, d\theta\, d\varphi = 2\pi a^2\int_0^{\pi}\sin\varphi\, d\varphi = 4\pi a^2$$

⚠ we don't need to include any additional factor here!

Surface area of the graph of a function:

$S: \quad z = f(x,y)$

i.e. $\quad \vec{r}(x,y) = \langle x, y, f(x,y) \rangle \quad \Longrightarrow \begin{cases} \vec{r}_x = \langle 1, 0, f_x(x,y) \rangle \\ \vec{r}_y = \langle 0, 1, f_y(x,y) \rangle \end{cases}$

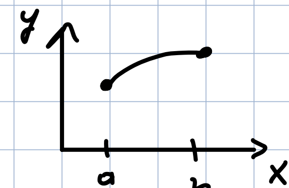
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

and $\quad |\vec{r}_x \times \vec{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$

$$\Longrightarrow \boxed{\operatorname{Area}(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA}$$

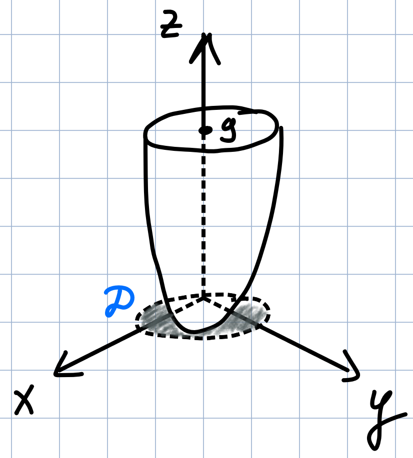
Rmk: Recall the arc length formula:

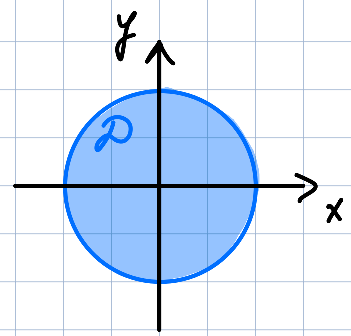
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Ex: $S: \begin{cases} z = x^2 + y^2 \\ z \le 9 \end{cases}$ Find Area(S).

Sol:





$D: \quad x^2 + y^2 \le 3^2$

$$\vec{r}(x,y) = \langle x, y, \overbrace{x^2+y^2}^{z(x,y)} \rangle$$

$$\frac{\partial z}{\partial x} = 2x \quad , \quad \frac{\partial z}{\partial y} = 2y$$

$$\text{Area}(S) = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \iint_D \sqrt{1 + 4(x^2+y^2)}\, dA$$

$$\underset{\text{coord.}}{\overset{\text{polar}}{=\!=}} \int_0^{2\pi} \int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta \underset{du = 2r\,dr}{\overset{u=r^2}{=\!=}} \int_0^{2\pi} \underbrace{\int_0^9 \sqrt{1+4u}\, \frac{1}{2} du}_{\frac{1}{8}\cdot\frac{2}{3}(1+4u)^{3/2}\Big|_{u=0}^{u=9}} d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}\left(37^{3/2} - 1\right) d\theta = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$